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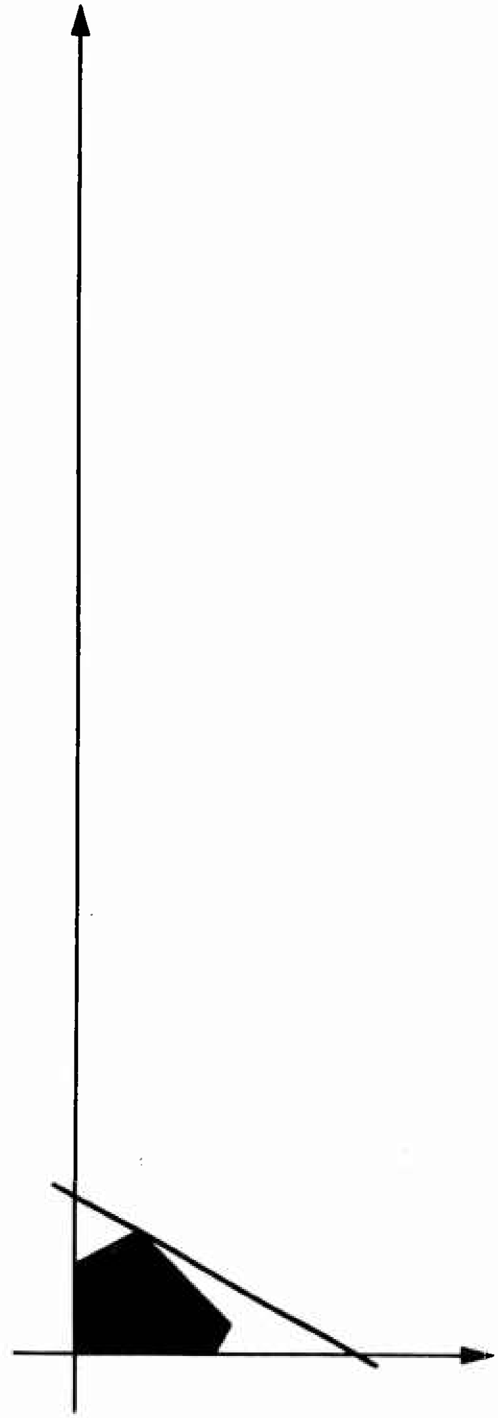

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SUPPORTS OF A CONVEX FUNCTION

Let C be a real, symmetric, mxm, positive-semi-definite matrix. Let $R^m = \left\{(x_1, \ldots, x_m) \mid x_i \text{ is a real number, } i = 1, \ldots, m\right\}$, and let $K \subset R^m$ be a polyhedral convex cone, i.e., there exists a real mxn matrix A such that $K = \left\{x \mid x \in R^m \text{ and } xA \leq 0\right\}$. Consider the function $\psi : K \rightarrow R$ defined by $\psi(x) = (xCx^T)^{1/2}$ for all $x \in K$. We wish to characterize the set, U, of all supports of ψ, where

$$(1) \qquad U = R^m \cap \left\{u \mid x \in K \Rightarrow ux^T \leq (xCx^T)^{1/2}\right\}.$$

Let $R^n_+ = R^n \cap \left\{\pi \mid \pi \geq 0\right\}$ and consider the set

$$(2) \qquad V = \left\{v \mid \exists\, x \in R^m, \ \pi \in R^n_+ \text{ and } v = \pi A^T + xC, \ xCx^T \leq 1, \ xA \leq 0\right\}.$$

We shall demonstrate:

THEOREM:

$$U = V.$$

We first show:

LEMMA 1

$$x, y \in R^m \Rightarrow (xCy^T)^2 \leq (xCx^T)(yCy^T).$$

Proof: If $x, y \in R^m$ consider the polynomial $p(\lambda) = \lambda^2 xCx^T + 2\lambda xCy^T + yCy^T = (x+\lambda y)C(x+\lambda y)^T$. Since C is positive-semi-definite, $p(\lambda) \geq 0$ for all real numbers λ, and thus the discriminant of p is non-positive, i.e.,

$$4(xCy^T)^2 - 4(xCx^T)(yCy^T) \leq 0 \ . \qquad \text{q.e.d.}$$

As an immediate application of Lemma 1 we show:

LEMMA 2

$$V \subset U$$

Proof: Let $v \in V$, then there exist $x \in R^m$, $\pi \in R^n_+$ such that $v = \pi A^T + xC$, $xCx^T \leq 1$. Now if $y \in R^m$, $yA \leq 0$, then $vy^T = yA\pi^T + xCy^T$ and $vy^T \leq xCy^T$, because $yA \leq 0$, $\pi^T \geq 0$ and $yA\pi^T \leq 0$. Thus, $vy^T \leq (xCx^T)^{1/2}(yCy^T)^{1/2}$, by Lemma 1, and $vy^T \leq (yCy^T)^{1/2}$, because $xCx^T \leq 1$. Thus, $v \in U$.

q.e.d.

From the fact that C is positive-semi-definite, it follows that:

LEMMA 3

The set V is convex.

Proof: If $x_k \in R^m$, $\pi_k \in R^n_+$, $x_k A \leq 0$, $u_k = \pi_k A^T + x_k C$, $x_k C x_k^T \leq 1$, $\lambda_k \in R_+$ for $k = 1, 2$ and $\lambda_1 + \lambda_2 = 1$, then: $\lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1 \pi_1 + \lambda_2 \pi_2)A^T + (\lambda_1 x_1 + \lambda_2 x_2)C$, $(\lambda_1 x_1 + \lambda_2 x_2)A \leq 0$, $\lambda_1 x_1 + \lambda_2 x_2 \in R^m$, $\lambda_1 \pi_1 + \lambda_2 \pi_2 \in R^n_+$, and $(\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - 1 \leq$

$$\leq (\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - \lambda_1 x_1 C x_1^T - \lambda_2 x_2 C x_2^T =$$

$$= -\lambda_1 \lambda_2 \left[x_1 C x_1^T - 2x_1 C x_2^T + x_2 C x_2^T \right] =$$

$$= -\lambda_1 \lambda_2 (x_1 - x_2)C(x_1 - x_2)^T \leq 0,$$ because C is positive-semi-definite.

q.e.d.

LEMMA 4

The set V is closed.

Proof: Let $\{w_k\}$ be a sequence with $w_k \epsilon R^m$, $k = 1, 2, \ldots$. We define the (pseudo) norm of w_k, denoted $\|\{w_k\}\|$, to be the smallest non-negative integer p such that there exists a k_o and for all $k \geq k_o$, x_k has at most p nonzero components. Now, suppose u is in the closure of V, i.e., there exist sequences $\{u_k\}$, $\{\pi_k\}$ and $\{x_k\}$ such that

(3)
$$\left.\begin{array}{l} \pi_k \epsilon R^n_+ , \; x_k \epsilon R^m , \; u_k = \pi_k A^T + x_k C \\ x_k A \leq 0 \quad \text{and} \quad y_k C x_k^T \leq 1 , \end{array}\right\} \quad k = 1, 2, \ldots$$

and $\{u_k\}$ converges to u.

Suppose the sequence $\{x_k\}$ is bounded, then we may assume, having taken an appropriate subsequence, that for some $x \epsilon R^m$, $\{x_k\} \to x$ and thus, by (3), $xA \leq 0$ and $xCx^T \leq 1$. Now, $yA \leq 0 \Longrightarrow u_k y^T - x_k C y^T = \pi_k A^T y^T = yA\pi_k^T \leq 0$, all $k \Longrightarrow uy^T - xCy^T \leq 0$. Thus the system,

$$y \epsilon R^m$$
$$yA \leq 0$$
$$(u - xC)y^T > 0$$

has no solution and by the usual feasibility theorem for linear inequalities (see e.q. (4) or (5)) the system:

$$\pi \in R_+^n$$

$$\pi A^T = u - xC$$

has a solution, and thus $u \in V$.

We have just demonstrated that if $\{x_k\}$ is bounded, then $u \in V$. Since $|\{x_k\}| + |\{x_k A\}| \leq m+n$, it is always possible to choose $\{x_k\}$ and $\{\pi_k\}$ satisfying (3) and such that $|\{x_k\}| + |\{x_k A\}|$ is minimal. We shall show next that if $\{x_k\}$, $\{\pi_k\}$ are so chosen, then $\{x_k\}$ is indeed bounded, thus completing the proof. Suppose then that $\{x_k\}$ is not bounded, i.e., $|x_k| = (x_k x_k^T)^{1/2} \to \infty$, and we may assume that $|x_k| > 0$ for all k. Let

$$z_k = \frac{x_k}{|x_k|}, \qquad k = 1, 2, \ldots$$

then $\{z_k\}$ is bounded and we may assume that there is a $z \in R^m$ such that the z_k converge to z and $|z| = 1$. From (3) it follows that $z_k A \leq 0$ and $z_k C z_k^T \leq \frac{1}{|x_k|}$ for all k. Thus, $zA \leq 0$ and $zCz^T \leq 0$. But then, from Lemma 1, $zCy^T = 0$ for all $y \in R^m$, and $zC = 0$. Summarizing:

(4) $$z \in R^m, \ zA \leq 0, \ zC = 0 .$$

Note that if z has a nonzero component, then infinitely many x_k's must have the same component nonzero, this follows from the fact that z is the limit of $\frac{x_k}{|x_k|}$. As a consequence, if $\{\lambda_k\}$ is any sequence of real numbers, then $|\{x_k + \lambda_k z\}| \leq |\{x_k\}|$. If $zA \neq 0$, and a^j, $j = 1, \ldots, n$, denotes the j^{th} column of A, let

$$\lambda_k = \max_{\substack{j \\ za^j < 0}} - \frac{x_k a^j}{za^j} .$$

Then we may replace, in (3), x_k by $x_k + \lambda_k z$ because $\lambda_k za^j + x_k a^j \leq 0$ for all j, and $(x_k + \lambda_k z)A \leq 0$, also $zC = 0$ and thus $(x_k + \lambda_k z)C = x_k C$, $(x_k + \lambda_k z)C(x_k + \lambda_k z)^T = x_k C x_k^T \leq 1$. However each $(x_k + \lambda_k z)A$ has at least one more zero component than $x_k A$, contradicting the minimality of $\left|\left\{x_k\right\}\right| + \left|\left\{x_k A\right\}\right|$. Thus, $zA = 0$ and we may replace, in (3), x_k by $x_k + \lambda_k z$ for an <u>arbitrary sequence</u> $\left\{\lambda_k\right\}$. But $z \neq 0$ and we can define λ_k so that $x_k + \lambda_k z$ has at least one more zero component than x_k has, thus $\left|\left\{x_k + \lambda_k z\right\}\right| < \left|\left\{x_k\right\}\right|$. However, $(x_k + \lambda_k z)A = x_k A$, and $\left|\left\{(x_k + \lambda_k z)A\right\}\right| = \left|\left\{x_k A\right\}\right|$, contradicting the minimality assumption.

q.e.d.

Lastly, we show:

<u>LEMMA 5</u>

$$U \subset V$$

<u>Proof</u>: Suppose $u \notin V$. By Lemmas 3 and 4 V is a closed convex set, hence there is a hyperplane which separates u <u>strongly</u> from V (see [4]). Thus there exist $x \in R^m$ and $\alpha \in R$ such that

$$ux^T > \alpha \geq vx^T \qquad \text{all} \quad v \in V .$$

Now, if $\pi \in R_+^n$ then $v = \pi A^T$ is in V (taking $x = 0$ in the definition of V). Thus $xA\pi^T = \pi A^T x^T \leq \alpha$ for all $\pi \in R_+^n$, and $xA \leq 0$, $x \in K$. Also $v = 0$ is in V, so that $\alpha \geq 0$. If $u \in U$ then $0 \leq \alpha < ux^T \leq (xCx^T)^{1/2}$, thus $xCx^T > 0$ and

$$v = \frac{xC}{(xCx^T)^{1/2}} \quad \epsilon \; V ,$$

consequently,

$$(xCx^T)^{1/2} > \alpha \geq \frac{xCx^T}{(xCx^T)^{1/2}} = (xCx^T)^{1/2}$$

a contradiction. Thus $u \notin U$. q. e. d.

Note: A direct application of Lemmas 2 and 5 yields the theorem stated at the beginning.

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